

# Robust entropy expansiveness implies generic domination.

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March, 16, 2009

## Abstract

Let  $f : M \rightarrow M$  be a  $C^r$ -diffeomorphism,  $r \geq 1$ , defined on a compact boundaryless  $d$ -dimensional manifold  $M$ ,  $d \geq 2$ , and let  $H(p)$  be the homoclinic class associated to the hyperbolic periodic point  $p$ . We prove that if there exists a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$  such that for every  $g \in \mathcal{U}$  the continuation  $H(p_g)$  of  $H(p)$  is entropy-expansive then there is a  $Df$ -invariant dominated splitting for  $H(p)$  of the form  $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$  where  $E$  is contracting,  $G$  is expanding and all  $F_j$  are one dimensional and not hyperbolic.

2000 Mathematics Subject Classification: 37D30, 37C29, 37E30

## 1 Introduction

In this paper we study what are the consequences at the dynamical behavior of the tangent map  $Df$  of a diffeomorphism  $f : M \rightarrow M$ , assuming that  $f$  is robustly entropy expansive. In this direction we obtain that the tangent bundle has a  $Df$ -invariant dominated splitting of the form  $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$  where  $E$  is contracting,  $G$  is expanding and all  $F_j$  are one dimensional and not hyperbolic.

Let  $M$  be a compact connected boundary-less Riemannian  $d$ -dimensional manifold,  $d \geq 2$ , and  $f : M \rightarrow M$  a homeomorphism. Let  $K$  be a compact invariant subset of  $M$  and  $\text{dist} : M \times M \rightarrow \mathbb{R}^+$  a distance in  $M$  compatible with its Riemannian structure. For  $E, F \subset K$ ,  $n \in \mathbb{N}$  and  $\delta > 0$  we say that  $E$   $(n, \delta)$ -spans  $F$  with respect to  $f$  if for each  $y \in F$  there is  $x \in E$  such that  $\text{dist}(f^j(x), f^j(y)) \leq \delta$  for all  $j = 0, \dots, n-1$ . Let  $r_n(\delta, F)$  denote the minimum cardinality of a set that  $(n, \delta)$ -spans  $F$ . Since  $K$  is compact  $r_n(\delta, F) < \infty$ . We define

$$h(f, F, \delta) \equiv \limsup_{n \rightarrow \infty} \frac{1}{n} \log(r_n(\delta, F))$$

and the topological entropy of  $f$  restricted to  $F$  as

$$h(f, F) \equiv \lim_{\delta \rightarrow 0} h(f, F, \delta).$$

The last limit exists since  $h(f, F, \delta)$  increases as  $\delta$  decreases to zero.

**Definition 1.1.** For  $x \in K$  let us denote

$$\Gamma_\epsilon(x, f) \equiv \{y \in M / d(f^n(x), f^n(y)) \leq \epsilon, n \in \mathbb{Z}\}.$$

We will simply write  $\Gamma_\epsilon(x)$  instead of  $\Gamma_\epsilon(x, f)$  when it is understood which  $f$  we refer to.

Following Bowen (see [Bo]) we say that  $f/K$  is entropy-expansive or  $h$ -expansive for short, if and only if there exists  $\epsilon > 0$  such that

$$h_f^*(\epsilon) \equiv \sup_{x \in K} h(f, \Gamma_\epsilon(x)) = 0.$$

**Theorem 1.1.** [Bo, Theorem 2.4] For all homeomorphism  $f$  defined on a compact invariant set  $K$  it holds

$$h(f, K) \leq h(f, K, \epsilon) + h_f^*(\epsilon) \text{ in particular } h(f, K) = h(f, K, \epsilon) \text{ if } h_f^*(\epsilon) = 0.$$

A similar notion to  $h$ -expansiveness, albeit weaker, is the notion of *asymptotically  $h$ -expansiveness* introduced by Misiurewicz [Mi]: let  $K$  be a compact metric space and  $f : K \rightarrow K$  an homeomorphism. We say that  $f$  is asymptotically  $h$ -expansive if and only if

$$\lim_{\epsilon \rightarrow 0} h_f^*(\epsilon) = 0.$$

Thus, we do not require that for a certain  $\epsilon > 0$ ,  $h_f^*(\epsilon) = 0$  but that  $h_f^*(\epsilon) \rightarrow 0$  when  $\epsilon \rightarrow 0$ . It has been proved by Buzzi, [Bu], that any  $C^\infty$  diffeomorphism defined on a compact manifold is asymptotically  $h$ -expansive. The interested reader can found examples of diffeomorphisms that are not entropy expansive neither asymptotically entropy expansive in [Mi, PaVi].

Next we recall the notion of dominated splitting.

**Definition 1.2.** We say that a compact  $f$ -invariant set  $\Lambda \subset M$  admits a dominated splitting if the tangent bundle  $T_\Lambda M$  has a continuous  $Df$ -invariant splitting  $E \oplus F$  and there exist  $C > 0$ ,  $0 < \lambda < 1$ , such that

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \forall x \in \Lambda, n \geq 0. \quad (1)$$

Observe that if the topological entropy of a map  $f : M \rightarrow M$  vanishes,  $h(f) = 0$ , then automatically  $f$  is  $h$ -expansive. For instance Morse-Smale diffeomorphisms  $\varphi : M \rightarrow M$  are  $h$ -expansive. We remark that Morse-Smale diffeomorphisms are  $C^1$ -stable under perturbations and so they constitute a class which is robustly  $h$ -expansive.

Here we are interested in diffeomorphisms that exhibit a chaotic behavior, i.e.: their topological entropy is positive. Moreover, we restrict our study to homoclinic classes  $H(p)$  associated to saddle-type hyperbolic periodic points. Recall that the homoclinic class  $H(p)$

of a saddle-type hyperbolic periodic point  $p$  of  $f \in \text{Diff}^1(M)$  is the closure of the intersections between the unstable manifold  $W^u(p)$  of  $p$  and the stable manifold  $W^s(p)$  of  $p$ . These classes persist under perturbations and we wish to establish the property of those classes under the assumption that  $h$ -expansiveness is robust.

**Definition 1.3.** *Let  $M$  be a compact boundaryless  $C^\infty$  manifold and  $f : M \rightarrow M$  be a  $C^r$  diffeomorphism,  $r \geq 1$ . Let  $H(p)$  be a  $f$ -homoclinic class associated to the  $f$ -hyperbolic periodic point  $p$ . Assume that there is a  $C^r$  neighborhood  $\mathcal{U}$  of  $f$ , such that for any  $g \in \mathcal{U}$  it holds that the continuation  $H(p_g)$  of  $H(p)$  is  $h$ -expansive. Then we say that  $f/H(p)$  is  $C^r$ -robustly  $h$ -expansive.*

In [PaVi, Theorem B] we obtain that if  $H(p, f)$  is isolated and the finest dominated splitting on  $H(p, f)$  is

$$T_{H(p,f)}M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$$

with  $E$  contracting,  $G$  expanding and all  $F_j$ ,  $j = 1, \dots, k$ , one dimensional and not hyperbolic, then  $f/H(p, f)$  is  $h$ -expansive. Moreover, since the dominated splitting is preserved under  $C^1$ -perturbations this result holds for a  $C^1$ -neighborhood  $\mathcal{U}(f) \subset \text{Diff}^1(M)$ , i.e.:  $h$ -expansiveness is  $C^1$ -robust.

Roughly speaking, [PaVi, Theorem B] says that the domination property implies that small neighbourhoods in  $H(p)$  have an ‘ordered dynamics’ and there cannot appear ‘arbitrarily small horseshoes’, i.e., horseshoes generated by homoclinic points in  $W_\xi^s(x) \cap W_\xi^u(x)$  for  $\xi > 0$  arbitrarily small and  $x \in H(p)$  periodic, as in the example given in [PaVi][Section 2] for a surface diffeomorphism. The presence of these arbitrarily small horseshoes would imply that  $\sup_{x \in H(p)} h(f, \Gamma_\epsilon(x)) > 0$  for any  $\epsilon > 0$ .

This paper is intended to continue [PaVi] in the reverse direction: we analyze the consequences of  $h$ -expansiveness to hold in a  $C^1$ -neighbourhood  $\mathcal{U}(f) \subset \text{Diff}^1(M)$  of  $f$ . Our main results are the following:

**Theorem A.** *Let  $M$ ,  $f : M \rightarrow M$  and  $H(p)$  be as in Definition 1.3 for  $r = 1$ . Then  $H(p)$  has a dominated splitting  $E \oplus F$ .*

In fact [PaVi, Example 2] shows that in dimension greater or equal to three the existence of a dominated splitting for  $H(p)$  is not enough to guarantee  $h$ -expansiveness, so it is natural to search for a stronger property.

Let us recall the concept of *finest dominated splitting* introduced in [BDP].

**Definition 1.4.** *Let  $\Lambda \subset M$  be a compact  $f$ -invariant subset such that  $TM/\Lambda = E_1 \oplus E_2 \oplus \cdots \oplus E_k$  with  $E_j$   $Df$  invariant,  $j = 1, \dots, k$ . We say that  $E_1 \oplus E_2 \oplus \cdots \oplus E_k$  is dominated if for all  $1 \leq j \leq k - 1$*

$$(E_1 \oplus \cdots \oplus E_j) \oplus (E_{j+1} \oplus \cdots \oplus E_k)$$

has a dominated splitting. We say that  $E_1 \oplus E_2 \oplus \cdots \oplus E_k$  is the finest dominated splitting when for all  $j = 1, \dots, k$  there is no possible decomposition of  $E_j$  as two invariant sub-bundles having domination.

An improvement of Theorem A is the following.

**Theorem B.** *Let  $M$ ,  $f : M \rightarrow M$  and  $H(p)$  be as in Definition 1.3 for  $r = 1$ . Then the finest dominated splitting in  $H(p)$  has the form  $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$  where all  $F_j$  are one dimensional and not hyperbolic.*

If  $H(p)$  is isolated then we may refine the previous result. Before we announce precisely this result, let us recall the definitions of: chain recurrent set, isolated homoclinic class and heterodimensional cycles..

**Definition 1.5.** *The chain recurrent set of a diffeomorphism  $f$ , denoted by  $R(f)$ , is the set of points  $x$  such that, for every  $\epsilon > 0$ , there is a closed  $\epsilon$ -pseudo orbit joining  $x$  to itself: there is a finite sequence  $x = x_0, x_1, \dots, x_n = x$  such that  $\text{dist}(f(x_i), x_{i+1}) < \epsilon$ .*

**Definition 1.6.** *We say that  $H(p)$  is isolated if there are neighborhoods  $\mathcal{U}$  of  $f$  in  $\text{Diff}^1(M)$  and  $U$  of the homoclinic class  $H(p)$  in  $M$  such that, for every  $g \in \mathcal{U}$ , the continuation  $H(p_g)$  of  $H(p)$  coincides with the intersection of the chain recurrence set of  $g$ ,  $R(g)$  with the neighborhood  $U$ .*

**Remark 1.2.** *Generically a recurrence class which contains a periodic point  $p_g$  coincides with  $H(p_g)$ , [BC].*

**Definition 1.7.** *We say that  $\Gamma$  is a cycle if  $\Gamma = \{p_i, 0 \leq i \leq n, p_0 = p_n\}$ , where  $p_i$  are hyperbolic periodic points of  $f$  and  $W^u(p_i) \cap W^s(p_{i+1}) \neq \emptyset$ , for all  $0 \leq i \leq n-1$ .  $\Gamma$  is called a heterodimensional cycle if, for some  $i \neq j$ ,  $\dim(W^u(p_i)) \neq \dim(W^u(p_j))$ .*

Recall that the *index* of a hyperbolic periodic point  $p$  is the dimension of its unstable manifold  $W^u(p)$ .

**Theorem C.** *Let  $M$ ,  $f : M \rightarrow M$  and  $H(p)$  be as in Definition 1.3 for  $r = 1$ . Assume moreover that  $f/H(p)$  is isolated. Then for  $g$  in  $\mathcal{U}(f)$ ,  $H(p_g)$  has a dominated splitting of the form  $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$  where  $E$  is contracting,  $G$  is expanding and all  $F_j$  are not hyperbolic and  $\dim(F_j) = 1$ . Moreover, in case that the index of periodic points in  $H(p_g)$  are in a  $C^1$  robust way equal to  $\text{index}(p)$  then for an open dense subset  $\mathcal{V} \subset \mathcal{U}(f)$ ,  $H(p_g)$  is hyperbolic, i.e.:  $k = 0$ .*

On the other hand, if there are  $g$  arbitrarily  $C^1$ -close to  $f$  such that in  $H(p_g)$  there are periodic points of different index then  $H(p)$  is approximated by robust heterodimensional cycles, [BDi].

If we do not assume that  $H(p)$  is isolated but we know that  $f$  cannot be approximated by  $g$  exhibiting a heterodimensional cycle we have the following result:

**Theorem D.** *Let  $\mathcal{C}(M) = \{f \in \text{Diff}^1(M); f \text{ has no cycles}\}$ , and  $H(p)$  be as in Definition 1.3 for  $r = 1$ . Assume that  $f \in \text{Diff}^1(M) \setminus \overline{\mathcal{C}(M)}$ . Then for  $g$  in a residual subset  $\mathcal{R} \subset \mathcal{U}(f)$ ,  $H(p_g)$  has a dominated splitting of the form  $E^s \oplus E^c \oplus E^u$  where  $E^c$  is not hyperbolic and  $\dim(E^c) \leq 2$ ,  $E^s$  is contracting and  $E^u$  is expanding. Moreover, if  $\dim(E^c) = 2$  then  $E^c = E_1^c \oplus E_2^c$  dominated.*

### 1.1 Idea of the proofs

The proofs of Theorems A and B go by contradiction: under the hypothesis that there is not a dominated splitting in  $T_{H(p)}M$ , we profit from some ideas of [PV] and [Ro] to create a flat tangency between  $W^s(p)$  and  $W^u(p)$ . We remark that in [PV, Ro] for the case that  $\dim(M) > 2$  it was proved that if  $r \geq 2$  and  $g$  has a homoclinic tangency then there are diffeomorphisms arbitrarily  $C^r$ -close to  $g$  exhibiting persistent homoclinic tangencies (thus generalizing results of [Nh1], see also [Nh2]). In our case, since we can perform the perturbations in the  $C^1$  topology, our arguments are simpler than theirs to obtain a  $C^2$  diffeomorphism  $g$  exhibiting a flat tangency, and afterward create an arc of tangencies between  $W^s(p)$  and  $W^u(p)$ .

Next we follow [DN], to perform another  $C^1$ -perturbation with support in a small neighborhood of the arc of tangencies leading to the appearance of arbitrarily small horseshoes with positive entropy contradicting  $h$ -expansiveness. Therefore  $Df/T_{H(p,f)}M$  admits a dominated splitting.

Moreover, either the finest dominated splitting (see Definition 1.4) has the form  $E \oplus F_1 \oplus \dots \oplus F_c \oplus G$  where all  $F_j$  are one dimensional and not hyperbolic or again we contradict robustness of  $h$ -expansiveness using [Go, Theorem 6.6.8].

For the proof of Theorem C we assume some specific generic properties described in Section 3 and that  $H(p)$  is isolated. These allow to prove that the extremal sub-bundles  $E$  and  $G$  are respectively contracting and expanding. Moreover if the index of periodic points of  $H(p_g)$  is robustly the index of  $p$  then for an open dense subset of  $\mathcal{U}(f)$  the dominated splitting defined on  $T_{H(p)}M$  is hyperbolic. This proof is done in two steps: (1) First we prove in Lemma 3.2 that the extremal sub-bundles are hyperbolic using the fact that  $H(p)$  is isolated, [BDPR]. (2) Second we show in Lemma 3.3 that if in a  $C^1$ -robust way the index of periodic points in  $H(p_g)$  are the same for  $g \in \mathcal{U}(f)$  then for an open and dense subset  $\mathcal{U}_1$  of  $\mathcal{U}(f)$  we have that  $H(p_g)$  is hyperbolic.

Finally in Theorem D, where we do not assume that  $H(p)$  is isolated, we see, under the generic assumptions described at Section 3, that for a residual subset  $\mathcal{R} \subset \mathcal{U}(f)$  we have a dominated splitting  $E^s \oplus E^c \oplus E^u$  defined on  $T_{H(p)}M$  such that  $E^s$  is contracting,  $E^u$  is expanding and  $E^c$  is dominated and at most two dimensional. For this we assume further that  $f \in \text{Diff}^1(M) \setminus \overline{\mathcal{C}(M)}$  which allows to use [Cr, MainTheorem].

## 2 Entropy expansiveness implies domination.

In this section we prove Theorem B assuming that  $f/H(p)$  is robustly  $h$ -expansive.

Let  $H(p)$  be a  $f$ -homoclinic class associated to the hyperbolic periodic point  $p$ . Assume that there is a  $C^1$  neighborhood  $\mathcal{U}$  of  $f$  such that for any  $g \in \mathcal{U}$  it holds that there is a continuation  $H(p_g)$  of  $H(p)$  such that  $H(p_g)$  is  $h$ -expansive.

We may assume that  $p$  is a *hyperbolic fixed point* since  $f/H(p)$  is  $h$ -expansive if and only if  $f^m/H(p)$  is  $h$ -expansive. This follows from the fact that for any compact  $f$ -invariant set  $\Lambda$  we have that  $h(f^m, \Lambda) = m \cdot h(f, \Lambda)$  which implies that  $h(f^m, \Lambda) = 0 \iff h(f, \Lambda) = 0$ .

Let  $x \in W^s(p) \cap W^u(p)$  be a transverse homoclinic point associated to the periodic point  $p$ . We define  $E(x) \equiv T_x W^s(p)$  and  $F(x) \equiv T_x W^u(p)$ . Since  $p$  is hyperbolic we have that  $E(x) \oplus F(x) = T_x M$ . Moreover,  $E(x)$  and  $F(x)$  are  $Df$ -invariant, i.e.:  $Df(E(x)) = E(f(x))$  and  $Df(F(x)) = F(f(x))$ . Denote by  $H_t(p)$  the set of the transverse homoclinic points associated to  $p$ . Then, it can be proved that  $H(p) \equiv \overline{H_t(p)}$ . Here  $\overline{A}$  stands for the closure in  $M$  of the subset  $A \subset M$ . So if we prove that there is a dominated splitting for  $H_t(p)$  we are done since we can extend by continuity the splitting to the closure  $H(p)$ . Moreover, since  $C^2$ -diffeomorphisms are dense in the  $C^1$ -neighbourhood  $\mathcal{U}$  we may assume that  $f$  is of class  $C^2$  taking into account that we are assuming that  $h$ -expansiveness is  $C^1$ -robust.

We will use the following result proved in [Fr]:

**Lemma 2.1.** [Fr, Lemma 1.1] *Let  $M$  be a closed  $n$ -manifold,  $f : M \rightarrow M$  a  $C^1$  diffeomorphism, and  $\mathcal{U}(f)$  a given neighbourhood of  $f$ . Then, there exist  $\mathcal{U}_0(f) \subset \mathcal{U}(f)$  and  $\delta > 0$  such that if  $g \in \mathcal{U}_0(f)$ ,  $S = \{p_1, p_2, \dots, p_m\} \subset M$  is a finite set, and  $L_i, i = 1, \dots, m$  are linear maps,  $L_i : TM_{p_i} \rightarrow TM_{f(p_i)}$ , satisfying  $\|L_i - D_{p_i}g\| \leq \delta, i = 1, \dots, m$  then there is  $\tilde{g} \in \mathcal{U}(f)$  satisfying  $\tilde{g}(p_i) = g(p_i)$  and  $D_{p_i}\tilde{g} = L_i$ . Moreover, if  $U$  is any neighborhood of  $S$  then we may chose  $\tilde{g}$  so that  $\tilde{g}(x) = g(x)$  for all  $x \in \{p_1, p_2, \dots, p_m\} \cup (M \setminus U)$ .*

**Remark 2.2.** *The statement given there is slightly different from that above, but the proof of our statement is contained in [Fr].*

### 2.1 Existence of dominated splitting: proof of Theorem A.

Under the hypothesis of Theorem A, let us assume that  $f$  is of class  $C^r$ ,  $r \geq 2$  and prove that there is a dominated splitting for  $H_t(p)$ .

The proof goes by contradiction and it is done in several steps: (1) at Lemma 2.3 we perform a perturbation  $g$  of  $f$  exhibiting a homoclinic point  $x_g \in H(p_g)$  with small angle between  $W_{loc}^s(x_g, g)$  and  $W_{loc}^u(x_g, g)$ , (2) at Proposition 2.5 we perform another perturbation (that we still denote by  $g$ ) of  $f$  to create a tangency between  $E^s(x, g)$  and  $E^u(x, g)$ ,  $x \in H(p_g)$ , (3) at Proposition 2.6 through another perturbation of  $f$  we create an arc of flat tangencies  $\beta \subset H(p_g)$ , (4) finally in Subsection 2.1.1 we perform a sequence of perturbations of  $f$  leading to  $G$  near  $f$  presenting a sequence of two by two disjoint small horseshoes

$H_{\epsilon_n} \subset H(p_G)$ ,  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover, we can select the sequence  $\epsilon_n$  in such a way that none of them are a constant of  $h$ -expansiveness of  $G$ . Since the entropy of each of these small horseshoes is positive, we arrive to a contradiction to  $h$ -expansiveness of  $f$ .

To start, let us assume, by contradiction, that  $H_t(p)$  has no dominated splitting. Then, by [MPP, § 3.6 Proof of Theorem F] it holds

(AD) for all  $m \in \mathbb{Z}^+$  there exists  $x_m$  such that for all  $0 \leq n \leq m$ ,

$$\|Df^n|E(x_m)\| \cdot \|Df^{-n}|F(f^n(x_m))\| > 1/2,$$

**Lemma 2.3.** *Assume that (AD) holds. Then, given  $\gamma > 0$  and  $\epsilon > 0$  there is  $m > 0$  and  $g$  an  $\epsilon$ - $C^1$ -perturbation of  $f$  with a homoclinic point  $x_g$  associated to  $p_g$  such that the angle at  $x_g$  between  $W_{loc}^s(x_g, g)$  and  $W_{loc}^u(x_g, g)$  is less than  $\gamma$ .*

*Proof.* Arguing by contradiction let us assume that there is  $\gamma_0 > 0$  such that for all  $g$  in  $\mathcal{U}_0$  the angle at  $x_g$  between  $W_{loc}^s(x_g, g)$  and  $W_{loc}^u(x_g, g)$  is greater or equal than  $\gamma_0$ .

By hypothesis there exist vectors  $v_m \in F(x_m)$  and  $w_m \in E(x_m)$  with  $\|v_m\| = \|w_m\| = 1$  such that

$$\frac{\|Df^j(w_m)\|}{\|Df^j(v_m)\|} > \frac{1}{2}, \quad \forall j, 1 \leq j \leq m.$$

Take  $\epsilon > 0$  small such that any  $C^1$ - $\epsilon$ -perturbation of  $f$  gives a diffeomorphism  $g \in \mathcal{U}_0$  where  $\mathcal{U}_0$  is the  $C^1$ -neighborhood of  $f$  where we have  $h$ -expansiveness. Let  $\epsilon' > 0$  be such that any perturbation of the derivatives along a finite orbit of  $f$  can be realized via Lemma 2.1 by a  $C^1$ - $\epsilon$ -perturbation of  $f$ .

Let us define  $T_j : T_{f^j(x_m)}M \rightarrow T_{f^j(x_m)}M$  a linear map such that  $T_j|_{E(f^j(x_m))} = (1 + \epsilon')id$  and  $T_j|_{F(f^j(x_m))} = id$ ,  $j = 0, \dots, m$ . Note that  $T_j$  stretches  $E = T_{x_m}W_\epsilon^s(x_m, f)$  and left  $F = T_{x_m}W_\epsilon^u(x_m, f)$  unchanged. Let  $P : T_{x_m}M \rightarrow T_{x_m}M$  be a linear map satisfying  $P = id$  in  $E(x_m)$  and  $P = id + L$  in  $F(x_m)$  where  $L : F(x_m) \rightarrow E(x_m)$  is a linear map such that  $L(v_m) = \epsilon'w_m$  and  $\|L\| = \epsilon'$ . Finally define  $G_0 = T_1 \cdot Df_{x_m} \cdot P$ , and  $G_j = T_{j+1} \cdot Df_{f^j(x_m)}$  for  $j = 1, \dots, m-1$ . By Lemma 2.1 there exists a diffeomorphism  $g : M \rightarrow M$  such that  $g$  is  $\epsilon$ -near  $f$ , keeps the orbit of  $x_m$  unchanged for  $j = 0, 1, \dots, m$ , and such that  $Dg_{f^j(x_m)} = G_j$ . We may assume (and do) that the support of the perturbation does not cut a small neighborhood of  $p$ . It follows that  $x_m$  continues to be a homoclinic point of  $g$ . Moreover, we do not change  $E(f^j(x_m))$ ,  $j \in \mathbb{Z}$ , and  $F(f^j(x_m))$  is changed only for  $j \geq 0$ . Thus such bundles are the stable and unstable directions of a homoclinic point of a diffeomorphism  $g \in \mathcal{U}_0$ . We obtain that  $v_m \mapsto v_m + \epsilon'w_m = u$  and after  $m$  iterates we have  $u_m = Dg^m(u) = Dg^m(v_m + \epsilon'w_m) = Df^m(v_m) + (1 + \epsilon')^m Df^m(\epsilon'w_m)$ .

Given  $\epsilon' > 0$  we may find  $m > 0$  such that  $\epsilon'(1 + \epsilon')^m \geq 4 + 2/\gamma_0$  where  $\gamma_0 > 0$  is, by hypothesis of absurd, such that  $\angle(E(x), F(x)) > \gamma_0$  for all  $x \in H_t(p_g)$ ,  $g \in \mathcal{U}_0$ , where

$\angle(E(x), F(x))$  stands for the angle between  $E(x)$  and  $F(x)$ . With this choice of  $m$ , by [Ma2, Lemma II.10] we have

$$\begin{aligned} \|Df^m(v_m)\| &= \|u_m - (1 + \epsilon')^m Df^m(\epsilon' w_m)\| \geq \\ &\geq \frac{\gamma_0}{1 + \gamma_0} \|u_m\| \geq \frac{\gamma_0}{1 + \gamma_0} \left| \|\epsilon'(1 + \epsilon')^m Df^m(w_m)\| - \|Df^m(v_m)\| \right|. \end{aligned}$$

Dividing the inequality  $\|Df^m(v_m)\| \geq \frac{\gamma_0}{1 + \gamma_0} \left| \|\epsilon'(1 + \epsilon')^m Df^m(w_m)\| - \|Df^m(v_m)\| \right|$  by  $\frac{\gamma_0}{1 + \gamma_0} \|Df^m(v_m)\|$  and taking into account that by hypothesis

$$\frac{\|Df^m(w_m)\|}{\|Df^m(v_m)\|} > \frac{1}{2} \quad \text{and} \quad \epsilon'(1 + \epsilon')^m \geq 4 + 2/\gamma_0$$

we find

$$\frac{1 + \gamma_0}{\gamma_0} > \frac{\epsilon'(1 + \epsilon')^m}{2} - 1 > 1 + 1/\gamma_0 = \frac{1 + \gamma_0}{\gamma_0},$$

arriving to a contradiction. Hence  $\angle(Dg^m(u), w_m) < \gamma$ , proving Lemma 2.3.  $\square$

Let us recall the following result which may be found in [BDP, Lemma 4.16], see also [BDPR, Lemma 3.8].

**Theorem 2.4.** *Let  $p$  be a hyperbolic periodic point and  $H(p)$  its homoclinic class. Assume that  $H(p)$  is not trivial. Then there exists an arbitrarily small  $C^1$ -perturbation  $g$  of  $f$  and a hyperbolic periodic point  $q$  of  $H(p_g)$  with period  $\pi(q)$  and homoclinically related with  $p_g$  such that  $Df_q^{\pi(q)}$  has only positive real eigenvalues of multiplicity one.*

Observe that in the previous result, since  $q_g \in H(p_g)$ , we have  $H(p_g) = H(q_g)$ . So, to simplify notation, we may assume directly that  $p = q$  and moreover that  $g = f$ , and that  $p$  is a fixed point. We order the eigenvalues of  $Df_p$  labeling them as  $0 < \lambda_k < \dots < \lambda_1 < 1 < \mu_1 < \dots < \mu_{d-k}$  so that the less contracting and the less expanding ones are respectively  $\lambda_1$  and  $\mu_1$ .

By a small  $C^1$ -perturbations we may also assume that locally, in a neighborhood  $V$  of  $p$ , we have linearizing coordinates so that

$$f(x) = \sum_{j=1}^k \lambda_j a_j u_j + \sum_{j=1}^{d-k} \mu_j a_{k+j} u_{k+j}$$

where we write  $x = \sum_{j=1}^d a_j u_j$  for  $x \in V$ .

The lines in  $W_{loc}^s(p)/V$  corresponding to the eigenvalues  $\lambda_j$  may be extended to all of  $W^s(p)$  by backward iteration by  $f$  giving us a foliation by lines of dimension  $k$ . Similarly for  $W^u(p)$  we have a  $(d - k)$ -foliation by lines obtained by forward iteration by  $f$ .



Now, let us assume that  $g$  is near  $f$ ,  $f = g$  in a small neighborhood of  $p$  and that there is a small angle between  $T_x W^s(p, g)$  and  $T_x W^u(p, g)$  where  $x$  is a  $g$ -homoclinic point associated to  $p$ . That is: there is  $\gamma$  small such that

$$\angle(T_x W^s(p, g), T_x W^u(p, g)) < \gamma.$$

By Theorem 2.4, we may assume that all the eigenvalues of  $Df_p^{\pi_p}$  are positive with multiplicity one and that we have linearizing coordinates in a small neighborhood of  $p$ .

The next proposition establishes that if the angle between  $T_x W^s(p, g)$  and  $T_x W^u(p, g)$  is small than we can create a tangency between  $T_x W^s(p, \tilde{g})$  and  $T_x W^u(p, \tilde{g})$ , for some  $\tilde{g}$  near  $g$ .

**Proposition 2.5.** *There is  $\gamma > 0$  and  $\mathcal{U}_0(g) \subset \mathcal{U}(f)$  so that for some  $\tilde{g} \in \mathcal{U}_0(g)$  there is a tangency between  $E^s(x, \tilde{g})$  and  $E^u(x, \tilde{g})$  if  $\angle(E^s(x, g), E^u(x, g)) < \gamma$ . Moreover  $x$  is a homoclinic point of  $\tilde{g}$ ,  $E^s(x, \tilde{g}) \oplus E^u(x, \tilde{g})$  has dimension  $d - 1$  and there is  $N > 0$  so that if  $\langle u \rangle$  is the subspace common to  $E^s(x, \tilde{g})$  and  $E^u(x, \tilde{g})$  then  $(D\tilde{g})^N(\langle u \rangle)$  is tangent to the line corresponding to the less contracting eigenvalue and  $(D\tilde{g})^{-N}(\langle u \rangle)$  is tangent to the line corresponding to the less expanding eigenvalue of  $D_p \tilde{g}$ .*

*Proof.* Let  $\mathcal{U}(f)$ ,  $\mathcal{U}_0(f)$  and  $\delta$  be as in Lemma 2.1. Shrinking  $\mathcal{U}_0$  if it were necessary we may assume that  $\text{clos} \mathcal{U}_0(f) \subset \mathcal{U}(f)$ . Hence we may assume without loss of generality that there is some  $C > 0$  such that  $\sup\{\|D_x g\| : g \in \mathcal{U}_0(f)\} \leq C$ .

By hypothesis there is  $g \in \mathcal{U}_0(f)$ ,  $x \in W^s(p_g, g) \cap W^u(p_g, g)$  and  $\gamma > 0$  small so that

$$\angle(E^s(x, g), E^u(x, g)) < \gamma.$$

Taking  $\gamma < \delta/C$ , since  $\angle(E^s(x, g), E^u(x, g)) < \gamma$ , there exist  $v \in E^{s\perp}$  and  $u \in E^s$  such that  $v + u \in E^u$ ,  $\|u\| = 1$ ,  $\|v\| < \gamma$ . Let  $T : T_x M \rightarrow T_x M$  be such that  $T|_{E^{s\perp}} = 0$ ,  $T(u) = -v$  and  $\|T\| < \delta/C$ . Let  $L : T_{g^{-1}(x)} M \rightarrow T_x M$  be defined by  $L = (Id + T) \circ D_{g^{-1}(x)} g$ . Then we have

$$\|L - D_{g^{-1}(x)} g\| < \delta, \quad \text{and} \quad u \in L(E^u(g^{-1}(x))).$$

Take a neighborhood  $U$  of  $g^{-1}(x)$  such that  $\mathcal{O}_g(x) \cap U = \{g^{-1}(x)\}$ . Using Lemma 2.1 we find  $\tilde{g} \in \mathcal{U}(f)$  such that  $g^j(x) = \tilde{g}^j(x)$  for all  $j$ ,  $\tilde{g} = g$  outside  $U$ , and  $D_{g^{-1}(x)} \tilde{g} = L$ . Hence  $x \in W^s(p_{\tilde{g}}, \tilde{g}) \cap W^u(p_{\tilde{g}}, \tilde{g})$  since its forward and backward orbits continue to converge to  $p_{\tilde{g}}$ . Moreover  $u \in E^s(x, \tilde{g}) \cap E^u(x, \tilde{g})$  and so the intersection of  $W^s(p_{\tilde{g}})$  and  $W^u(p_{\tilde{g}})$  is not transverse at the point  $x$ .

Since the eigenvalues of  $Df_p$  are all real positive and of multiplicity one and  $f = g$  in a small neighborhood of  $p$ , by  $N$  forward iterations we have a vector  $D^N \tilde{g}(u)$  almost tangent to the straight line  $\langle v_1 \rangle$  corresponding to the less contracting eigenvalue at  $p$ . Again by Lemma 2.1 we can perturb  $\tilde{g}$  outside a small neighborhood of  $p$  to let the direction of  $(D\tilde{g})^N(u)$  coincide with  $\langle v_1 \rangle$ . Similarly we obtain  $(D\tilde{g})^{-N}(u)$  tangent to the line corresponding to the less expanding eigenvector of  $D\tilde{g}_p$ .  $\square$

From Proposition 2.5 we may assume for  $f$  itself that there is a homoclinic point of tangency  $x \in W^s(p) \cap W^u(p)$  with properties analogous to those of  $\tilde{g}$ . The next lemma asserts that under these hypothesis, we can obtain an arc  $\beta$  of non-transversal homoclinic points in  $W^s(p) \cap W^u(p)$ .

**Proposition 2.6.** *Let  $p$  be a hyperbolic fixed point for  $f$  of index  $k$  and  $x \in W^s(p) \cap W^u(p)$  such that the intersection at  $x$  is not transversal. Then by an arbitrarily small  $C^1$ -perturbation we may obtain a diffeomorphism  $g$  with  $x \in W^s(p_g, g) \cap W^u(p_g, g)$  such that the intersection at  $x$  is flat, there exists a small arc  $\beta$  contained in the intersection of the stable and unstable manifolds of  $p$ . Moreover, there is  $N > 0$  such that  $g^N(\beta) \subset W_{loc}^s(p, g)$  is tangent to the eigenvector corresponding to the less contracting eigenvalue and analogously  $g^{-N}(\beta) \subset W_{loc}^u(p, g)$  is tangent to the eigenvector corresponding to the less expanding eigenvalue.*

*Proof.* Since  $p$  is a hyperbolic saddle,  $W^s(p)$  is an Euclidean  $k$ -dimensional hyperplane and  $W^u(p)$  an Euclidean  $(d - k)$ -dimensional hyperplane both immersed in  $M$ . If the intersection at  $x$  of  $W^s(p)$  and  $W^u(p)$  is not transversal we should have a vector  $u \neq 0$  in  $T_x W^u(p) \cap T_x W^s(p)$ , i.e.: we have a tangency between  $W^s(p)$  and  $W^u(p)$  at the homoclinic point  $x$ . Using Lemma 2.1 we may assume that the subspace generated by  $u$  is the unique in common between  $T_x W^u(p)$  and  $T_x W^s(p)$ , that is  $T_x W^u(p) + T_x W^s(p)$  has dimension  $d - 1$ . Moreover, we also may assume that  $k \geq d - k$  (otherwise we may take  $f^{-1}$  instead of  $f$ ) and, again by Lemma 2.1, that the tangent space  $T_x W_\epsilon^u(x)$  intersects trivially  $(T_x W_\epsilon^s(x))^\perp$  the orthogonal complement of  $T_x W_\epsilon^s(x)$ . Under these assumptions the orthogonal projection of  $W_\epsilon^u(x)$  into  $W_\epsilon^s(x)$  is locally a diffeomorphism in a suitable neighborhood of  $x$ . Let us choose  $D_x \subset W_\epsilon^s(x)$  a small disk and  $N > 0$  such that  $f^N(D_x) \subset W_\epsilon^s(p)$ , and let  $L_x$  be a small disk in  $W_\epsilon^u(x)$  such that  $f^{-N}(L_x) \subset W_\epsilon^u(p)$ .  $L_x$  projects onto  $L'_x \subset D_x$  diffeomorphically. Via a local coordinate map we may identify  $D_x$  with

$$\{y \in \mathbb{R}^d / y_{k+1} = \dots = y_d = 0; y_1^2 + \dots + y_k^2 = 1\},$$

with  $x$  identified with the origin 0 and  $u$  having the direction of  $Oy_1$  which is tangent at 0 to  $L'_x$  too.  $L_x$  may be viewed as the graph of a map  $\Gamma : L'_x \rightarrow (T_x W_\epsilon^s(x))^\perp$  with  $\frac{\partial \Gamma}{\partial y_1}|_0 = 0$ . To simplify notation we write  $(y_1, \dots, y_k) = Y_1$  and  $(y_{k+1}, \dots, y_d) = Y_2$ . Hence if  $(Y_1, Y_2) \in L_x$  then  $Y_2 = \Gamma(Y_1(Z))$ , where, given  $L'_x$ ,  $Y_1(Z)$  is a local coordinate map from a neighborhood of 0 in  $\mathbb{R}^{d-k}$  to  $D_x$ .

**Claim 2.1.** *There exists a  $C^1$  perturbation of  $f$  that produces a diffeomorphism  $g \in \mathcal{U}(f)$  with a flat intersection at  $x \in D_x \cap L_x$ , with  $D_x \subset W_\epsilon^s(x)$  and  $L_x \subset W_\epsilon^u(x)$ . This flat intersection contains a small arc  $\beta$ .*

*Proof.* Define  $h : M \rightarrow M$  by

$$h(Y_1, Y_2) = (Y_1, Y_2 - G(Y_1, Y_2)\Gamma(y_1, 0, \dots, 0)).$$

Here  $G$  is a  $C^\infty$ -bump function,  $0 \leq G(Y_1, Y_2) \leq 1$ , that vanishes in the boundary of the ball  $B(0, \epsilon')$ , is equal to 1 in  $B(0, \epsilon'/4)$ , and such that  $\|\nabla G\| < \frac{2}{\epsilon'}$ , where  $\nabla$  means the gradient.

Let us see that  $h$  is a diffeomorphism  $\epsilon'$ - $C^1$ -close to the identity.

(a)  $h$  is injective: Indeed,  $h(Y_1, Y_2) = h(Y'_1, Y'_2)$  implies that  $Y_1 = Y'_1$ . Hence

$$Y_2 - G(Y_1, Y_2)\Gamma(y_1, 0, \dots, 0) = Y'_2 - G(Y_1, Y'_2)\Gamma(y_1, 0, \dots, 0).$$

Therefore

$$\|Y_2 - Y'_2\| = \|(G(Y_1, Y_2) - G(Y_1, Y'_2))\Gamma(y_1, 0, \dots, 0)\| \leq \|\Gamma(y_1, 0, \dots, 0)\|,$$

where we have used that  $0 \leq G(Z_1, Z_2) \leq 1$  for all  $(Z_1, Z_2)$ . Taking into account that

$$\Gamma(0, 0) = 0, \quad \frac{\partial \Gamma}{\partial y_1} \Big|_0 = 0$$

we obtain that  $\Gamma(y_1, 0, \dots, 0) = o(\epsilon')$ . Therefore

$$\|(G(Y_1, Y_2) - G(Y_1, Y'_2))\| = \langle \nabla G(Y_1, \Theta_2), Y_2 - Y'_2 \rangle \leq \|\nabla G\| \|\Gamma(y_1, 0, \dots, 0)\| < \frac{2}{\epsilon'} o(\epsilon').$$

Here  $(Y_1, \Theta_2)$  is a point in the segment joining  $(Y_1, Y_2)$  with  $(Y_1, Y'_2)$ . Let us choose  $\epsilon' > 0$  so small that  $\frac{2}{\epsilon'} \cdot o(\epsilon') < \frac{1}{2}$ . It follows that

$$\|Y_2 - Y'_2\| = \|(G(Y_1, Y_2) - G(Y_1, Y'_2))\Gamma(y_1, 0, \dots, 0)\| \leq \frac{1}{2} \|\Gamma(y_1, 0, \dots, 0)\|.$$

By induction we have that for all  $n \in \mathbb{N}$

$$\|Y_2 - Y'_2\| = \|(G(Y_1, Y_2) - G(Y_1, Y'_2))\Gamma(y_1, 0, \dots, 0)\| \leq \frac{1}{2^n} \|\Gamma(y_1, 0, \dots, 0)\|.$$

Therefore  $Y_2 = Y'_2$  and  $h$  is injective.

(b)  $h$  is a diffeomorphism: Indeed, we have

$$Dh = \begin{pmatrix} Id & \vdots & 0 \\ \dots & \vdots & \dots \\ -G \frac{\partial \Gamma}{\partial y_1}^t - \Gamma^t \frac{\partial G}{\partial Y_1} & \vdots & Id - \Gamma^t \frac{\partial G}{\partial Y_2} \end{pmatrix}$$

Here  $\Gamma = \Gamma(y_1, 0, \dots, 0)$ , analogously  $\frac{\partial \Gamma}{\partial y_1}$  only depends on  $y_1$ , and  $\Gamma^t$  is the transpose of  $\Gamma$ . As  $\frac{\partial \Gamma}{\partial y_1} \Big|_0 = 0$  we have that  $-G \frac{\partial \Gamma}{\partial y_1}^t$  is small if  $\epsilon'$  is sufficiently small and the same is true with respect to  $\Gamma^t \frac{\partial G}{\partial Y_1}$  and  $\Gamma^t \frac{\partial G}{\partial Y_2}$ , taking into account that  $\Gamma(y_1, 0, \dots, 0) = o(\epsilon')$  and  $\|\nabla G\| < \frac{2}{\epsilon'}$ . Thus  $Dh$  is invertible.

Items (a) and (b) above prove that  $h$  is a diffeomorphism as  $C^1$ -close to the identity map as we wish and  $h = id$  off a small ball  $B(x, \epsilon')$ . Now consider  $g = h \circ f$ . Then  $g$  is a small perturbation of  $f$ .

**Claim 2.2.**  *$x$  is a flat  $g$ -homoclinic point and there is an arc  $\beta \subset W^s(p, g) \cap W^u(p, g)$  with  $x \in \beta$ .*

Indeed, since  $x \in W^s(p, f) \cap W^u(p, f)$  we have that  $\lim_{n \rightarrow +\infty} f^n(x) = \lim_{n \rightarrow -\infty} f^n(x) = p$  and so  $x$  is neither forward recurrent nor backward recurrent. This implies that we may choose the support,  $B(x, \epsilon')$ , of the perturbation in such a way that for  $n \neq 0$ ,  $g^n(B(x, \epsilon')) \cap B(x, \epsilon') = \emptyset$ . Hence if  $y \in W_\epsilon^s(x, f)$  then for  $\epsilon > 0$  small we obtain that  $y \in W_\epsilon^s(x, g)$ . But  $h$  sends an arc  $\beta$  passing through  $x$  in  $W_\epsilon^u(x, f)$  onto an arc  $\gamma$  included in  $W_\epsilon^s(x, f) = W_\epsilon^s(x, g)$  and passing through  $x$  too. Therefore  $g^{-1} = f^{-1} \circ h^{-1}$  sends the arc  $\gamma$  into  $\beta$  which iterated successively by  $f^{-1}$  converges to  $p$ . Hence  $\beta$  is an arc contained in *both* the local stable and unstable manifold of  $x$  which is contained in  $W^s(p, g) \cap W^u(p, g)$ . Thus  $\beta$  is an arc of flat intersection between  $W^s(p, g)$  and  $W^u(p, g)$ . This finishes both the proofs of Claim 2.2 and Claim 2.1.  $\square$

It is not difficult to see that this perturbation  $g$  may be done in such a way that for  $N > 0$  great enough  $g^N(\beta) \subset W_{loc}^s(p, g)$  is tangent to the eigenvector corresponding to the less contracting eigenvalue and analogously  $g^{-N}(\beta) \subset W_{loc}^u(p, g)$  is tangent to the eigenvector corresponding to the less expanding eigenvalue.

All together finishes the proof Proposition 2.6.  $\square$

### 2.1.1 Creating small horseshoes.

The previous result gives a diffeomorphism  $g$ ,  $C^1$ -near  $f$ , such that the intersection between  $W^u(p, g)$  and  $W^s(p, g)$ , in a local chart around  $x$  such that  $T_x W_\epsilon^s(x) \cap T_x W_\epsilon^u(x) = \langle u \rangle$ , contains a segment  $\beta = \{su : -\delta \leq s \leq \delta\}$ . Moreover,  $Dg^N u$  is tangent to the line corresponding to the less contracting eigenvector of  $Dg_p$  and  $Dg^{-N} u$  is tangent to the line corresponding to the less expanding eigenvector of  $Dg_p$ .

Next we shall do a perturbation of  $g$ , which will give a diffeomorphism  $G$  such that  $G$  coincides with  $g$  outside a small neighborhood of  $\beta$ , similar to those of [DN, Lemma 5.1, Lemma 6.3] in order to create a sequence of small horseshoes  $H_n \subset H(p, G)$  associated to  $W_{loc}^s(x, G)$  and  $W_{loc}^u(x, G)$ . These horseshoes will have positive topological entropy and will be built in such a way that neither  $\epsilon > 0$ , nor  $\epsilon/2, \epsilon/4, \dots, \epsilon/2^n, \dots$  will be constants of  $h$ -expansiveness for  $H(p, G)$ . Therefore the diffeomorphism  $G$  is not  $h$ -expansive, contradicting our hypothesis.

To do so we proceed as follows: first, since we are working in a  $C^1$ -neighborhood of  $f$  and  $C^r$ ,  $r \geq 2$ , diffeomorphisms are dense in  $\text{Diff}^1(M)$  we may assume that  $g$ , the diffeo-

morphism obtained at Proposition 2.6, is of class  $C^r$ ,  $r \geq 2$ . We split the proof into two cases, according to the index of  $p$ .

## 2.2 $\text{index}(p) = d - 1$

Let us assume first that  $p$  is of index  $d - 1$ , i.e.:  $\dim(W^u(p, f)) = 1$ . This will simplify the techniques involved. We may assume, taking a large positive iterate by  $g$  and possibly reducing  $\delta$ , that  $\beta$ , the segment of tangency, is contained in the local stable manifold of  $p$  in a local chart which is a linearizing neighborhood  $U(p)$  of  $p$ .

Let  $\psi : [0, \delta] \rightarrow \mathbb{R}$  be a  $C^\infty$  bump function satisfying:

1.  $\psi(s) = 1/5$ , for  $s \in [0, \delta/16]$ . This implies that  $\psi^{(k)}(0) = \psi^{(k)}(\delta/16) = 0$  for all  $k \geq 1$ .
2.  $\psi'(s) < 0$  for  $s \in (\delta/16, \delta/8)$ .
3.  $\psi(s) = 0$  for all  $s \in [\delta/8, \delta/4]$ , this implies that  $\psi^{(k)}(\delta/8) = \psi^{(k)}(\delta/4) = 0$  for all  $k \geq 1$ .
4.  $\psi'(s) > 0$  for  $s \in (\delta/4, 3\delta/8)$ .
5.  $\psi(s) = 1$  for all  $s \in [3\delta/8, \delta]$ , this implies that  $\psi^{(k)}(3\delta/8) = \psi^{(k)}(\delta) = 0$  for all  $k \geq 1$ .

Next, consider  $b : (-\delta, 5\delta/4] \rightarrow \mathbb{R}$  such that

$$b(s) = \psi(s) \text{ for all } s \in [0, \delta],$$

$$b(s) = \frac{1}{5}\psi(2(s + \delta/2)) \text{ for all } s \in [-\delta/2, 0],$$

$$b(s) = \frac{1}{5^2}\psi(2^2(s + 3\delta/4)) \text{ for all } s \in [-3\delta/4, -\delta/2],$$

and in general

$$b(s) = \frac{1}{5^n}\psi(2^n(s + \delta(1 - 1/2^n))) \text{ for all } s \in [-\delta(1 - 1/2^n), -\delta(1 - 1/2^{n-1})].$$

Put also

$$b(s) = 5\psi\left(\frac{s - \delta}{2}\right) \text{ for } s \in [\delta, 5\delta/4].$$

It is easy to see that  $b(s)$  is  $C^\infty$  at  $(-\delta, 5\delta/4]$ . We may assume that for  $s \in [0, \delta]$ ,  $|b'(s)| \leq 24/\delta$  and  $|b''(s)| \leq K/\delta^2$ , for some  $K > 0$ .

Hence for  $s \in [-\delta(1 - 1/2^n), -\delta(1 - 1/2^{n-1})]$  we have

$$|b'(s)| = \frac{1}{5^n}2^n \left| \psi'\left(2^n\left(s + \frac{2^n - 1}{2^n}\delta\right)\right) \right| \leq \frac{24 \cdot 2^n}{5^n \delta}$$

and

$$|b''(s)| = \frac{4^n}{5^n} \left| \psi''(2^n(s + \frac{2^n - 1}{2^n}\delta)) \right| \leq \frac{4^n K}{5^n \delta^2}.$$

Therefore  $|b'(s)| \rightarrow 0$  and  $|b''(s)| \rightarrow 0$  when  $s \rightarrow -\delta$ . Setting  $b(-\delta) = 0$  we have that  $b'(-\delta) = b''(-\delta) = 0$  and  $b$  is of class  $C^2$  on  $[-\delta, 5\delta/4]$ .

Let  $w$  be the unit vector in  $T_x M$  tangent to the expanding eigenvector of  $Dg_p$ . Recall we are assuming that  $\dim(W^u(p, G)) = 1$ . Then  $w$  is not contained in  $T_x W^s(x, g) + T_x W^u(x, g)$  since  $T_x W^u(x, g)$  is tangent to  $T_x W^s(x, g)$ . Recall that  $(0, s, 0)$  are the coordinates of  $\beta$  in a local chart and that the interval  $(0, [-\delta, 5\delta/4], 0)$  is totally contained in  $\beta$ . In the plane given by the origin 0 (identified with  $x$ ) and the vectors  $u$  and  $w$  we consider the graph of the function  $\hat{l} : [\delta/4, 5\delta/4] \rightarrow \mathbb{R}$  given by

$$\hat{l}(s) = \epsilon_1 \cdot (s - \delta/2)(\delta - s), \quad s \in [\delta/4, 5\delta/4].$$

Observe that for  $s \in [\delta/4, 5\delta/4]$ ,  $\hat{l}(s)$  vanishes at  $s = \delta/2$  and  $s = \delta$  and it has a maximum value equals to  $\delta^2 \epsilon_1 / 16$  at  $s = 3\delta/4$ . Now we extend  $\hat{l}$  to  $[-\delta, 5\delta/4]$  in the following way:

$$\hat{l}(s) = \epsilon_2 \cdot (s + \delta/4)(-s), \quad s \in [-3\delta/8, \delta/8],$$

$$\hat{l}(s) = \epsilon_3 \cdot (s + 5\delta/8)(-\delta/2 - s), \quad s \in [-11\delta/16, -7\delta/16],$$

and in general for  $n \geq 1$ :

$$\hat{l}(s) = \epsilon_{n+1} \cdot (s + \delta(1 - 3/2^{n+1}))(-\delta(1 - 1/2^{n-1}) - s), \quad s \in [-\delta(1 - 5/2^{n+2}), -\delta(1 - 9/2^{n+2})].$$

For  $s \in [-\delta(1 - 5/2^{n+2}), -\delta(1 - 9/2^{n+2})]$ ,  $\hat{l}$  vanishes only at  $s_{n_1} = -\delta(1 - 3/2^{n+1})$  and  $s_{n_2} = -\delta(1 - 1/2^{n-1})$  and it has a maximum value  $\delta^2 \epsilon_{n+1} / (5^n \cdot 2^{2n+4})$  at  $(s_{n_1} + s_{n_2})/2$ . We complete the definition of  $\hat{l}$  in  $[-\delta, 5\delta/4]$  setting  $\hat{l}(s) = 0$  elsewhere.

Finally, let  $l(s) = \hat{l}(s)b(s)$  for all  $s \in [-\delta, 5\delta/4]$ . Then  $l(s)$  is  $C^\infty$  in  $(-\delta, 5\delta/4]$  and  $C^2$  in  $[-\delta, 5\delta/4]$ .

Put coordinates in the local chart  $Y = (S, s, t)$  and denote by  $B_s$  a small  $(d - 1)$ -dimensional disk around  $x$  contained in a fundamental domain of  $W_{loc}^s(p, g)$  whose coordinates in the local chart are  $(S, s, 0)$ . Analogously denote by  $B_u$  a small 1-dimensional disk contained in  $W^u(p, g)$  around  $x$  whose coordinates in the local chart are  $(0, s, 0)$ . Note that  $B_s$  is characterized by  $t = 0$ ; and  $B_u$  is the arc  $\beta$  contained in  $B_s$ , parameterized by  $s \in [-\delta, 5\delta/4]$ . The point  $x$  is identified with  $(0, 0, 0)$ .

Now, pick another  $C^\infty$  bump function  $\varphi$  such that  $\varphi$  vanishes outside a  $\epsilon$  neighborhood of  $\beta$ ,  $\epsilon \geq 2\epsilon_1$ , and is equal to 1 in the  $\epsilon/2$  neighborhood of  $\beta$ .

Let  $h : M \rightarrow M$  be given by

$$(S, s, t) \mapsto (S, s, t + l(s)\varphi(\|Y\|))$$

and  $h = id$  outside  $B(\beta, \epsilon)$  where  $\epsilon$  is such that the  $\epsilon$ -neighborhood of  $\beta$  does not intersect  $U \cap g(U) \cap g^{-1}(U)$ .

Now, letting  $G = h \circ g$ , we get, by construction, that  $G$  is a small perturbation of  $g$ , and, as in Proposition 2.6, it is not difficult to see that  $B_s \subset W_{loc}^s(x, G) \subset W^s(p, G)$  and  $(0, s, l(s)) \subset W_{loc}^u(x, G) \subset W^u(p, G)$ . Furthermore, it is straightforward to show that  $W^s(p, G)$  and  $W^u(p, G)$  intersect transversely at the points

$$(0, \delta/2, 0), (0, \delta, 0), (0, -\delta/4, 0), (0, 0, 0), \dots, (0, -\delta(1-3/2^{n+1}), 0), (0, -\delta(1-1/2^{n-1}), 0), \dots$$

and the absolute value of the tangent of the angles at the points

$$(0, -\delta(1-3/2^{n+1}), 0), (0, -\delta(1-1/2^{n-1}), 0) \quad \text{is} \quad \frac{\epsilon_{n+1}\delta}{5^n 2^{n+1}}, \quad n \in \mathbb{N}.$$

We denote by  $\beta'$  the graph of  $l(s)$  in the plane  $0uw$ . If we choose  $\epsilon$ ,  $\epsilon_1 \geq \epsilon_2 \geq \dots \geq \epsilon_n \geq \dots$  with  $\epsilon_n \searrow 0$  and  $\delta$  small, we may obtain the perturbation  $G = h \circ g$  to be  $C^1$  small (see [Nh1]). Moreover, we can also assume that :

1.  $G = g$  on  $U \cap g(U) \cap g^{-1}(U)$ , where we recall that  $U = U(p)$  is a linearizing neighborhood of  $p$ .
2.  $W_{loc}^s(p, g) = W_{loc}^s(p, G)$  and  $W_{loc}^u(p, g) = W_{loc}^u(p, G)$ . Here  $loc > 0$  states for a suitable small positive number,
3.  $W_{loc}^s(x, G) \cup W_{loc}^u(x, G) \subset U \setminus G(U)$ . In particular  $\beta \cup \beta' \subset U \setminus G(U)$ .
4.  $G^k(W_{loc}^s(x, G)) \subset U$  for all  $k \geq 0$  and there is  $T > 0$  such that  $G^{-k}(W_{loc}^u(x, G)) \subset U$  for all  $k \geq T$ ,
5.  $G^{-T}(\beta \cup \beta') \subset U \setminus G^{-1}(U)$ .

We point out that item (5) above follows from the fact that we may reduce the value of  $\delta$ , if it were necessary, in order to ensure it.

**Lemma 2.7.** *There exists a sequence  $\epsilon_n \searrow 0$  such that  $G$  is not  $h$ -expansive.*

*Proof.* Recall that we are working in a linearizing neighborhood  $U$  of  $p$  with respect to  $g$ . Set

$$U_k^u = U \cap g(U) \cap \dots \cap g^k(U) \quad \text{and} \quad U_k^s = U \cap g^{-1}(U) \cap \dots \cap g^{-k}(U).$$

Let  $\gamma' = G^{-T}(\beta') \subset U \setminus G^{-1}(U)$  and denote by  $(0, 0, d_0)$ ,  $(0, 0, d_\infty)$  the coordinates of the end points of  $\gamma'$  corresponding respectively to  $s = 5\delta/4$  and  $s = -\delta$ . In the same way we label all points in  $\gamma'$  corresponding to the *transverse* intersections of  $\beta$  with  $\beta'$ :  $(0, 0, d_1)$  corresponds to  $(0, \delta/2, 0)$  and  $(0, 0, d'_1)$  corresponds to  $(0, \delta, 0)$ ,  $(0, 0, d_2)$  corresponds to  $(0, -\delta/4, 0)$  and  $(0, 0, d'_2)$  corresponds to  $(0, 0, 0)$ ,  $(0, 0, d_3)$  corresponds to  $(0, -5\delta/8, 0)$  and

$(0, 0, d'_3)$  corresponds to  $(0, -\delta/2, 0)$ , and so on, labeling the image by  $G^{-T}$  of all the points of transverse intersection between  $\beta$  and  $\beta'$ .

Take small arcs  $a_1^s$  and  $a_1'^s$  contained in  $U \setminus G^{-1}(U)$  tangent to the the direction of the eigenvector corresponding to the weakest contracting eigenvalue of  $(DG)_p$  at the points  $(0, 0, d_1)$  and  $(0, 0, d'_1)$ . Multiply them by a  $(d-2)$ -dimensional disk  $C$  of diameter  $c$ . Analogously take small arcs  $a_1^u$  and  $a_1'^u$  tangent to the direction corresponding to the eigenvector of the expanding eigenvalue of  $(DG)_p$  at the points  $(0, \delta/2, 0)$  and  $(0, 0, d'_1)$  and contained in  $U \setminus G(U)$ . By the  $\lambda$ -lemma, [PdeM][Lemma 7.1], the forward orbits of  $a_1^u$  and  $a_1'^u$  contain arcs arbitrarily  $C^1$  near  $W^u(p, G)$  and the backward orbits of  $a_1^s \times C$  and  $a_1'^s \times C$  contain  $(d-1)$ -dimensional disks arbitrarily  $C^1$  near  $W^s(p, G)$ . By the way we have chosen  $a_1^s$  and  $a_1'^s$  and the assumption about the eigenvalues of  $D(G)_p$  (all positive real), we have that there is  $k_1 = k_1(\epsilon_1, \delta)$  such that for  $k \geq k_1$  in  $U$  we have  $\text{dist}(G^{-k}(a_1^s), \beta) < \epsilon_1 \delta^2/32$  and  $\text{dist}(G^{-k}(a_1'^s), \beta) < \epsilon_1 \delta^2/32$ . Moreover, we may choose  $c > 0$  small such that  $G^{-k}(a_1^s \times C)$  and  $G^{-k}(a_1'^s \times C)$  cut  $\beta'$  but is contained in the  $\epsilon/4$  neighborhood of  $\beta$  and therefore  $\varphi = 1$  there.

In the local coordinates we have chosen, we pick a thin rectangle  $R_1$  with top and bottom given by  $G^{-k_1}(a_1^s \times C)$  and  $G^{-k_1}(a_1'^s \times C)$  and bounded in its sides by segments parallel to the  $w$ -axis which is transverse to  $D_S$ . Increasing  $k_1$  and reducing  $c$ ,  $a_1^s$  and  $a_1'^s$ , if it were necessary, we may assume that  $G^{k_1}(R)$  is contained in the  $c$ -neighborhood of the graph of  $\beta'$  restricted to  $[3\delta/8, 9\delta/8]$ .

Set  $g_1 = G^{k_1}$  and let  $g_2 = G^T|(U \setminus G^{-1}(U)) : (U \setminus G^{-1}(U)) \rightarrow (U \setminus G(U))$  and consider

$$\Lambda_1 = \bigcap_{n \in \mathbb{Z}} (g_2 \circ g_1)^n(R_1).$$

Then  $\Lambda_1$  contains a horseshoe  $H_1$  (see [Nh1, DN]) and therefore  $H_{\epsilon_1} = \bigcup_{j=0}^{k_1+T-1} G(H_1)$  has positive topological entropy. Since this horseshoe is arbitrarily small we may assume that there is a periodic point  $p_1 \in H_1$  such that  $H_1 \subset \Gamma_\epsilon(p_1)$  see Definition 1.1, where  $0 < 2\epsilon_1 \leq \epsilon$ . Moreover, the periodic point  $p_1$  is homoclinically related to  $p$  since by the  $\lambda$ -lemma we have that positive iterates by  $(g_2 \circ g_1)^{-1}$  give thin subrectangles crossing all of  $R_1$  and hence the stable manifold of  $p_1$  cuts  $W_{loc}^u(x) \subset W^u(p, G)$  and analogously positive iterates by  $g_2 \circ g_1$  gives subrectangles close to  $\beta'$  in the Hausdorff metric and therefore the unstable manifold of  $p_1$  cuts  $W_{loc}^s(x) \subset W^s(p, G)$ .

**Claim 2.3.** *There is  $\{\epsilon_n\}_{n=1}^\infty$  such that for every  $\epsilon_n$  it is associated a horseshoe  $H_{\epsilon_n}$  with  $H_{\epsilon_n} \subset H(p, G)$  and  $\lim_{n \rightarrow \infty} \text{diam}(H_{\epsilon_n}) = 0$ .*

*Proof.* Let us choose  $\epsilon_2 > 0$  and construct  $H_{\epsilon_2}$ . For this, pick  $\epsilon_2 \leq \epsilon_1$  such that  $G^{-k_1}(a_1^s \times C)$  and  $G^{-k_1}(a_1'^s \times C)$  are at a distance greater than  $\epsilon_2$  from  $(S, s, 0)$ . Since  $\epsilon_n \leq \epsilon_2$  for all  $n \geq 2$  we have that no part of the graph of  $l(s)$  for  $s \in [-\delta, \delta/4]$  cuts  $R_1$ .



We found a new rectangle  $R_2$  disjoint from  $R_1$  contained in  $U_{k_2}^s \setminus U_{k_2+1}^s$  with  $k_2 > k_1$  applying again the  $\lambda$ -Lemma. Increasing  $k_2$  and reducing the corresponding values of  $c_2$ ,  $a_2^s$  and  $a_2'^s$ , if it were necessary, we may assume that  $G^{k_2}(R_2)$  is contained in the  $c_2$ -neighborhood of the graph of  $\beta'$  restricted to  $[-5\delta/16, \delta/16]$ . By construction when we iterate by  $G$  the images of  $R_1$  and  $R_2$  cannot intersect since in  $U \setminus G(U)$  there are only one iterate of  $R_1$  and one iterate of  $R_2$  (namely  $R_1$  and  $R_2$ ). We then have for  $G$  two disjoint small horseshoes,  $H_1, H_2$  both with periodic points  $p_1, p_2$  homoclinically related to  $p$  (the proof that  $p_2$  is homoclinically related to  $p$  is the same than that to  $p_1$ ). Hence both  $H_1$  and  $H_2$  are included in  $H(p, G)$ .

Next we choose  $\epsilon_3 \leq \epsilon_2 \leq \epsilon_1$  so that  $G^{-k_2}(a_2^s \times C_2)$  and  $G^{-k_2}(a_2'^s \times C_2)$  are at a distance greater than  $\epsilon_3$  from  $(S, s, 0)$ . For such  $\epsilon_3$ , there is a horseshoe  $H_{\epsilon_3}$  disjoint from  $H_{\epsilon_1}$  and  $H_{\epsilon_2}$  but still contained in  $H(p, G)$ . This construction follows the same steps as before: first find a thin rectangle  $R_3$  cutting the graph of  $l(s)$  only for  $s \in [-21\delta/32, -15\delta/32]$ ,  $R_3 \cap R_1 = \emptyset$ ,  $R_3 \cap R_2 = \emptyset$ . Then find an appropriate positive real number  $k_3 > k_2$  such that  $G^{k_3}(R_3)$  is contained in the  $c_3$ -neighborhood of the graph of  $\beta'$  restricted to  $[-21\delta/32, -15\delta/32]$ .

In this way we may pick the sequence  $\epsilon_n$  such that for every  $n$  it is associated a horseshoe  $H_{\epsilon_n}$  satisfying (1)  $\lim_{n \rightarrow \infty} \text{diam}(H_{\epsilon_n}) \rightarrow 0$ , (2)  $H_{\epsilon_j} \cap H_{\epsilon_i} = \emptyset$  and (3)  $H_{\epsilon_n} \subset H(p, G)$  for all  $n \in \mathbb{Z}^+$ . This proves Claim 2.3.  $\square$

Since the topological entropy of  $H_{\epsilon_n}$  is positive for all  $n$ , and  $H_{\epsilon_n} \subset H(p, G)$ , we conclude that  $G/H(p, G)$  is not  $h$ -expansive, violating robustness of  $h$ -expansiveness. The proof of Lemma 2.7 is complete.  $\square$

Then, the final conclusion is that hypothesis (AD) described in the begining of this section can not hold. In another words, we conclude that there exists  $m > 0$  such that for all homoclinic point  $x \in H(p)$  there is  $1 \leq k \leq m$  such that

$$\|Df^k/E(x)\| \|Df^{-k}/F(f^k(x))\| \leq \frac{1}{2}.$$

Following [SV, Theorem A], it can be built a dominated splitting for the homoclinic points of  $H(p, f)$  as required, and then extend it by continuity to the whole  $H(p, f)$  using that the closure of the homoclinic points coincide with  $H(p, f)$ .

Thus in the case of  $p$  a periodic point of index  $d - 1$  the proof of Theorem A follows.

**Remark 2.8.** *Let us point out that even though we can assume that  $g$ , the diffeomorphism with a segment of homoclinic tangencies, is  $C^\infty$ , the bump function  $l(s)$ , used to perturb it, is just  $C^2$ . Hence it seems that a similar construction can be used to prove the stronger result that  $G/H(p)$  is not asymptotically  $h$ -expansive. Recall, [Bu, BFF], that  $C^\infty$ -diffeomorphisms are asymptotically  $h$ -expansive so that a  $C^\infty$  perturbation of a  $C^\infty$  diffeomorphism does not disprove asymptotically  $h$ -expansiveness.*

### 2.2.1 $\text{index}(p) = k < d = \dim(M)$

For the general case of  $\text{index}(p) = k < d = \dim(M)$  the proof is similar, the perturbation  $h$  of  $g$  given  $G = h \circ g$  has to be adapted as we sketch below.

Let  $w$  be the unit vector in  $T_x M$  tangent to the less expanding eigenvector of  $Dg_p$ . Then  $w$  is not contained in  $T_x W^s(x, g) + T_x W^u(x, g)$  see Propositions 2.5 and 2.6. In a local chart around  $x$ ,  $(0, s, 0)$  represent the coordinates of the arc  $\beta$  but the coordinates  $(S, s, T)$  are such that  $S$  is a  $(k - 1)$ -dimensional vector, and  $T$  a  $(d - k)$  dimensional vector that we split as  $(t, T') = T$  with  $t$  one-dimensional. As in the codimension one case we have that  $(0, [-\delta, 5\delta/4], 0, 0)$  is totally contained in  $\beta$ . In the plane given by the origin 0 (identified with  $x$ ) and the vectors  $u$  corresponding to  $(0, 1, 0, 0)$  and  $w$  corresponding to  $(0, 0, 1, 0)$  we, as above, consider the graph of the function  $\hat{l} : [\delta/4, 5\delta/4] \rightarrow \mathbb{R}$  given by

$$\hat{l}(s) = \epsilon_1 \cdot (s - \delta/2)(\delta - s), \quad s \in [\delta/4, 5\delta/4].$$

Now we extend  $\hat{l}$  to  $[-\delta, 5\delta/4]$  and define the  $C^2$ -function  $l(s)$  as in the codimension one case.

Put coordinates in the local chart  $Y = (S, s, t, T)$  and denote by  $B_s$  a small  $k$ -dimensional disk around  $x$  contained in a fundamental domain of  $W_{loc}^s(p, g)$  whose coordinates in the local chart are  $(S, s, 0, 0)$ . Analogously denote by  $B_u$  a small  $d - k$ -dimensional disk contained in  $W^u(p, g)$  around  $x$  whose coordinates in the local chart are  $(0, s, 0, T)$ . Note that  $B_s$  is characterized by  $t = 0, T = 0$ ; and  $B_u$  contains the arc  $\beta$  contained in  $B_s$ , parameterized by  $s \in [-\delta, 5\delta/4]$ . The point  $x$  is identified with  $(0, 0, 0, 0)$ .

Now, pick a  $C^\infty$  bump function  $\varphi$  such that  $\varphi$  vanishes outside a  $\epsilon$  neighborhood of  $\beta$ ,  $\epsilon \geq 2\epsilon_1$ , and is equal to 1 in the  $\epsilon/2$  neighborhood of  $\beta$ .

Let  $h : M \rightarrow M$  be given by

$$(S, s, t, T) \mapsto (S, s, t + l(s)\varphi(\|Y\|), T)$$

and  $h = id$  outside  $B(\beta, \epsilon)$  where  $\epsilon$  is such that the  $\epsilon$ -neighborhood of  $\beta$  does not intersect  $U \cap g(U) \cap g^{-1}(U)$ .

Now, letting  $G = h \circ g$ , we get, by construction, that  $G$  is a small perturbation of  $g$ , and, as in Proposition 2.6, it is not difficult to see that  $B_s \subset W_{loc}^s(x, G) \subset W^s(p, G)$  and  $(0, s, l(s), T) \subset W_{loc}^u(x, G) \subset W^u(p, G)$ .

The remaining of the proof of Theorem A follows in a similar way to that of the codimension one case.

## 3 Proof of Theorems B and C

In this section we prove both Theorems B and C. For this, let us first remark that after [ABCDW, §2.1],  $C^1$ -generically the finest dominated splitting has a very special form. Thus, before we continue, let us first put  $f$  in that context.

*Generic assumptions.* There exists a residual subset  $\mathcal{G}$  of  $\text{Diff}^1(M)$  such that if  $f : M \rightarrow M$  is a diffeomorphism belonging to  $\mathcal{G}$  then

1.  $f$  is Kupka-Smale, (i.e.: all periodic points are hyperbolic and their stable and unstable manifolds intersect transversally)
2. for any pair of saddles  $p, q$ , either  $H(p, f) = H(q, f)$  or  $H(p, f) \cap H(q, f) = \emptyset$ .
3. for any saddle  $p$  of  $f$ ,  $H(p, f)$  depends continuously on  $g \in \mathcal{G}$ .
4. The periodic points of  $f$  are dense in  $\Omega(f)$ .
5. The chain recurrent classes of  $f$  form a partition of the chain recurrent set of  $f$ .
6. every chain recurrent class containing a periodic point  $p$  is the homoclinic class associated to that point.

Taking into account [Go, Corollary, 6.6.2, Theorem 6.6.8], that guarantees that the homoclinic tangency can be associated to a saddle inside the homoclinic class, the next result is proved in [ABCDW, Corollary 3]:

**Theorem 3.1.** (*[ABCDW, Corollary 3]*) *There is a residual subset  $\mathcal{I} \subset \mathcal{G}$  of  $\text{Diff}^1(M)$  such that if  $f \in \mathcal{I}$  has a homoclinic class  $H(p, f)$  which contains hyperbolic saddles of indices  $i < j$  then either*

1. *For any neighborhood  $U$  of  $H(p, f)$  and any  $C^1$ -neighborhood  $\mathcal{U}$  of  $f$  there is a diffeomorphism  $g \in \mathcal{U}$  with a homoclinic tangency associated to a saddle of the homoclinic class  $H(p_g, g)$ , where  $p_g$  is the continuation of  $p$ . or*
2. *There is a dominated splitting*

$$T_{H(p, f)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G$$

*with  $\dim(E) = i$  and  $\dim(F_h) = 1$  for all  $h$  and  $\dim(G) = \dim(M) - j$ . Moreover, the sub-bundles  $F_h$  are not hyperbolic.*

**Proof of Theorem B.**

Let  $H(p) \subset M$  be a homoclinic class robustly entropy expansive, i.e., there is a neighbourhood  $\mathcal{U} \subset \text{Diff}^1(M)$  such that  $f \in \mathcal{U}$ , there is a continuation  $H(p_g)$  of  $H(p)$  for all  $g \in \mathcal{U}$  and  $H(p_g)$  is  $h$ -expansive. By Theorem A we have a dominated splitting defined on  $T_{H(p)}M$ . Moreover, by [Go, Theorem 6.6.8], we have that in  $H(p_g)$  there is a finest dominated splitting which has the form

$$T_{H(p_g, g)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G \tag{2}$$

with  $E$ ,  $G$  and  $F_h$   $Df$ -invariant sub-bundles,  $h = 1, \dots, j - i$ , and all  $F_h$  one-dimensional, and

$$E \prec F_1 \prec F_2 \cdots \prec F_{j-i} \prec G.$$

Otherwise, by the theorem of [Go] cited above, we may create with an arbitrarily small  $C^1$ -perturbation a tangency *inside* the perturbed homoclinic class. After that we repeat the arguments of 2.1.1 contradicting  $h$ -expansiveness. Theorem B is proved.

**Proof of Theorem C.** By [CMP] there is residual subset  $\mathcal{R}_0$  of  $\text{Diff}^1(M)$  such that, for every  $f \in \mathcal{R}_0$ , any pair of homoclinic classes of  $f$  are either disjoint or coincide. For  $f \in \mathcal{R}_0$ , by [Ab], the number of different homoclinic classes of  $f$  is locally constant in  $\mathcal{R}_0$ . We split the proof into two cases: (1) this number is finite (and in this case  $f$  is *tame*) or (2) there are infinitely many distinct homoclinic classes (and in this case  $f$  is *wild*).

**$f$  is tame** In this case  $H(p)$  is isolated. Before we continue, recall that if  $V \subset M$  and  $\Lambda_f(V)$  is the maximal invariant set of  $f$  in  $V$ , i.e.:  $\Lambda_f(V) = \bigcap_{n \in \mathbb{Z}} f^n(V)$ , then set  $\Lambda_f(V)$  is robustly transitive if there is a  $C^1$ -neighbourhood  $\mathcal{U}$  of  $f$  such that  $\Lambda_g(\overline{V}) = \Lambda_g(V)$  and  $\Lambda_g(V)$  is transitive for all  $g \in \mathcal{U}$  (i.e.:  $\Lambda_g(V)$  has a dense orbit).

**Lemma 3.2.** *Assume  $f : M \rightarrow M$  is tame and that  $T_{H(p)}M$  has a dominated splitting of the form (2). Then  $E$  is contracting and  $G$  is expanding.*

*Proof.* Since  $H(p)$  is isolated it is a robustly transitive set maximal invariant in a neighbourhood  $U \subset M$  and hence, according to [BDPR][Theorem D], the extremal sub-bundles  $E$  and  $G$  are contracting and expanding respectively.  $\square$

Under the same hypothesis of the previous lemma either we have that in a  $C^1$ -robust way the index of periodic points in  $H(p_g)$ ,  $g$  near  $f$ , are the same and equal to  $\text{index}(p)$  or there are  $g$  arbitrarily  $C^1$ -close to  $f$  such that in  $H(p_g)$  there are periodic points of different index. In the first case we have

**Lemma 3.3.** *There is a dense open subset  $\mathcal{U}_1$  of  $\mathcal{U}(f)$  in the  $C^1$  topology such that for all  $g \in \mathcal{U}_1$  we have that  $H(p_g)$  is hyperbolic.*

*Proof.* We follow the lines of the proof at [BDi, Section 6]. Since  $H(p)$  is isolated by [BC, Corollaire 1.13] or [Ab, Theorem A] it is robustly isolated. Let  $E$  and  $F$  be sub bundles such that  $T_{H(p_g)}M = E \oplus F$  is  $m$ -dominated, for all  $g \in \mathcal{U}(f)$ , with  $\dim(E) = \text{index}(p)$ . We need to prove that  $\|Df^n_{|E(x)}\| \rightarrow 0$  as  $n \rightarrow +\infty$  and  $\|Df^{-n}_{|F(x)}\| \rightarrow 0$  as  $n \rightarrow +\infty$  for any  $x \in H(p_g)$  in order to prove that  $H(p_g)$  is hyperbolic. Let us show only that  $\|Df^n_{|E(x)}\| \rightarrow 0$  as  $n \rightarrow +\infty$ , the other one being similar. For this, it is enough to show that for any  $x \in H(p_g)$  there exists  $k = k(x)$  such that  $\prod_{i=0}^k \|Dg^n_{|E(g^{im}(x))}\| < \frac{1}{2}$ .

Arguing by contradiction, assume this does not hold. Then, there exist  $z \in H(p_g)$  such that  $\prod_{i=0}^k \|Df^m_{|E(f^{im}(z))}\| \geq \frac{1}{2} \quad \forall k \geq 0$ .

As in the proof of [Ma2, Theorem B] we may find  $y \in H(p_g) \cap \Sigma(g)$ , where  $\Sigma(g)$  is a set of total probability measure, such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{g^{mi}(y)} g^m / E(g^{mi}(y))\| \geq 0$$

Thus there is a perturbation  $h$  of  $g$  such that  $h$  has a non hyperbolic periodic point in  $H(p_h)$ . After a new perturbation we obtain periodic points  $P$  and  $Q$  contained in a small neighborhood  $U$  of  $H(p_h)$  and with different indices. Since  $H(p)$  is  $C^1$ -robustly isolated  $P, Q \in H(p_h)$  contradicting our assumption that in a  $C^1$ -robust way the index of periodic points in  $H(p)$  are the same and equal to  $\text{index}(p)$ . This finishes the proof of Theorem C in this case.  $\square$

In the second case, that is, there are  $g$  arbitrarily  $C^1$ -close to  $f$  such that in  $H(p_g)$  there are periodic points of different indices, by [GW],  $C^1$ -generically the diffeomorphism  $g$ , and hence  $f$ , can be  $C^1$  approximated by diffeomorphisms exhibiting a heterodimensional cycle. Next we show that in this case the eigenvalues of periodic points are robustly in  $\mathbb{R}$ .

**Lemma 3.4.** *Let us assume that there is a periodic point  $q \in H(p)$  with expanding complex eigenvalues such that  $\text{index}(q) < \text{index}(p)$ . Then there is an arbitrarily  $C^1$ -small perturbation of  $f$  creating a tangency inside the perturbed homoclinic class  $H(p_g)$ .*

*Proof.*  $C^1$  generically we may assume that there is a robust heterodimensional cycle between  $p$  and  $q$  and that  $W^s(p) \cap W^u(q)$  contains a compact arc  $l$  homeomorphic to  $[0, 1]$ , (see [BDi]). Let us consider a disk  $D$  of the same dimension  $s$  of  $W^s(p)$  and contained in  $W^s(p)$  such that  $D$  is homeomorphic to  $[0, 1] \times [-1, 1]^{s-1}$  by a homeomorphism  $h$  such that  $h([0, 1] \times \{0\}^{s-1}) = l$ . Iterating by  $f^{-\pi(q)}$  this arc  $l$  spirals around  $q$  while  $D$  stretches approaching  $W^s(q)$ . Since  $W^s(q) \cap W^u(p) \neq \emptyset$  there is a  $C^1$  small perturbation of  $f$  creating a tangency between  $W^s(p_g)$  and  $W^u(p_g)$ .  $\square$

**Corollary 3.5.** *If there is a periodic point  $q \in H(p)$  with expanding complex eigenvalues such that  $\text{index}(q) < \text{index}(p)$  then  $H(p)$  is not  $C^1$  robustly  $h$ -expansive.*

*Proof.* Under the hypothesis of the lemma we may create tangencies inside  $H(p)$  and by another  $C^1$ -perturbation an arbitrarily small horseshoe in the intersection between  $W_{loc}^s(p)$  and  $W_{loc}^u(p)$  contradicting  $h$ -expansiveness.  $\square$

Thus Corollary 3.5 implies that the eigenvalues of periodic points in  $H(p)$  are real numbers in a robust way. By [ABCDW] for  $C^1$ -generic diffeomorphisms the set of indices of the (hyperbolic) periodic points in a homoclinic class form an interval in  $\mathbb{N}$ . Thus by [BDi][Theorem 2.1] there are diffeomorphisms arbitrarily  $C^1$ -close to  $f$  with  $C^1$ -robust heterodimensional cycles.

As a consequence we obtain in both cases the following result:

**Theorem 3.6.** *If  $f/H(p)$  is  $C^1$  robustly  $h$ -expansive and  $H(p)$  is an isolated homoclinic class then for a dense open subset  $\mathcal{U}' \subset \mathcal{U}(f)$  either  $f/H(p)$  is hyperbolic and we have  $T_{H(p)}M = E^s \oplus E^u$  or there is a robust heterodimensional cycle in  $H(p_g)$  for  $g$  arbitrarily close to  $f$ .*

*Proof.* If we have that in a  $C^1$ -robust way the index of periodic points in  $H(p_g)$  are the same and equal to  $\text{index}(p_g)$  by Lemma 3.3 there is an open dense subset  $\mathcal{V}$  of  $\mathcal{U}(g)$  such that  $H(p_g)$  is hyperbolic for  $g \in \mathcal{V}$ . Hence we are done. Otherwise we have an open subset  $\mathcal{U}(g)$  in any neighborhood  $\mathcal{V} \subset \mathcal{U}(f)$  of any diffeomorphism  $g \in \mathcal{U}(f)$  exhibiting a heterodimensional cycle, [BDi]. This finishes the proof Theorem 3.6, which in its turn gives the proof of Theorem C.  $\square$

**$f$  is wild** Now let us assume that  $H(p)$  is not isolated. Either there is a small  $C^1$ -perturbation  $g$  of  $f$  such that  $H(p_g)$  is isolated or  $H(p)$  is persistently not isolated, i.e.:  $H(p_g)$  is not isolated for any  $g$  close to  $f$ . In the first case we are done by Theorem 3.6.

In the second case the following result of [Cr] (see also [W]) is valid assuming that  $f$  is far from homoclinic cycles.

**Remark 3.7.** *Since  $f/H(p)$  is  $h$ -expansive we are far from homoclinic tangencies.*

**Theorem 3.8** (Crovisier). *There exists a dense  $G_\delta$  subset of  $\text{Diff}^1(M) \setminus \overline{\text{Tang} \cup \text{Cycles}}$  such that each homoclinic class  $H$  has a dominated splitting  $T_H M = E^s \oplus E_1^c \oplus E_2^c \oplus E^u$  which is partially hyperbolic and such that each central bundle  $E_1^c, E_2^c$  has dimension 0 or 1.*

Thus Theorem D is a consequence of Theorem 3.8 and the previous remark.

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